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Mixture-Ratio Control to Improve Hydrogen-Fuel Rocket Performance

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The rocket equation of motion, in the absence of gravitational and aerodynamic forces, is integrated by considering the propellant mixture ratio as the problem control variable. The theory of optimal control is applied to obtain the control law, which maximizes the payload for an assigned velocity increment when the tank mass is assumed to be proportional to the propellant volume. The control strategy, which minimizes the rocket dry mass for assigned payload and velocity increment, is also investigated. For a low-velocity increment, the mixture-ratio range required by the optimal control is not ample, and the improvement, with respect to the constant mixture-ratio operation, is small; however, large differences in the mean mixture ratio are present, depending on the performance index. The optimal control law is almost independent of the considered index for the most demanding missions, whereas the mixture ratio exhibits a large variation: in this case the control of the mixture ratio can significantly improve rocket performance with respect to the constant mixture-ratio operation.

Nomenclature

A, B, D, E	= parameters; see Eqs. (12), (13), (18), and (A1), respectively
c	= effective exhaust velocity
H	= Hamiltonian; see Eq. (5)
m	= mass
T	= thrust
t	= time
V	= velocity
α	= mixture ratio
ΔV	= characteristic velocity
ϵ	= structural efficiency
λ	= adjoint variable
ρ	= bulk density
φ	= performance index; see either Eq. (11) or Eq. (17)

Subscripts

c	= constant mixture-ratio operation
d	= dry mass
e	= engine or constant mass
f	= final value
h	= hydrogen
i	= initial value
max	= maximum value
mean	= mean value
min	= minimum value
o	= oxygen
p	= propellant
t	= tank mass
u	= useful mass
v	= variable mixture-ratio operation

Superscripts

$*$	= maximum effective exhaust velocity
$'$	= derivative with respect to α

Introduction

PRESENTLY, a single-stage-to-orbit (SSTO) launch system is considered as a means to minimize the hardware costs. Rocket propulsion has lower development costs compared to airbreathing propulsion, whereas vertical takeoff and landing (VTOL) concepts offer simpler body geometry and lower dynamic loads. A rocket-powered reusable VTOL vehicle is being demonstrated for NASA.[1] Tsiolkovsky's equation[2] makes it immediately evident that a substantial improvement of the current structure and engine technology would probably be required for the successful operation of the full-scale vehicle: any device that can possibly improve system performance deserves attention.

Tsiolkovsky's equation[2] is obtained by integrating the rocket equation in the absence of gravitational and aerodynamic forces, when a constant specific impulse (i.e., a constant propellant mixture ratio) is assumed. An ordinary differentiation (see Appendix) provides the best value of the constant mixture ratio, that is, the best compromise between engine performance and structural mass, when the tank mass is assumed to be proportional to the propellant volume. It is well known that vehicle performance can be improved by burning RP-1 and liquid hydrogen (LH2) with liquid oxygen (LOX) in tripropellant engines or by varying the mixture ratio in LOX/LH2 bipropellant engines.[3–5] A high-density specific impulse can be obtained during the first phase of the ascent trajectory, whereas the highest specific impulse is exploited in the final phase. Compared with tripropellant engines, variable mixture-ratio engines might be preferable as complexity and weight are reduced.

In their paper, Martin and Manski[5] analyze the ascent trajectory of a rocket whose engines can be operated with two different values of the mixture ratio. In this paper, a continuous variation of the mixture ratio is permitted, and the theory of optimal control (OCT) is applied to obtain the control law, which either maximizes the payload or minimizes the dry mass. For the sake of simplicity, the equation of motion of a LOX/LH2 rocket in free space is considered; the solution of this problem is rigorously independent of the thrust level, and the influence of the mixture ratio on the propellant flow rate can be neglected. The model, however, should be sufficient for a preliminary comparison of the performance that is obtained by operating the engines with the best variable or constant mixture ratio. More accurate analyses are required subsequently to account at least for the influence of the thrust level on the gravitational and aerodynamic losses during an ascent trajectory.

Statement of the Problem

In a vacuum, when a rocket with mass m is only subject to its thrust T, the equation of motion is

$$\frac{dV}{dt} = \frac{T}{m} = \frac{c}{m}\frac{dm_p}{dt} \tag{1}$$

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The mixture ratio α (oxidizer/fuel mass ratio) relates the propellant flow rate to the hydrogen flow rate:

$$\frac{\mathrm{d}m_p}{\mathrm{d}t} = -\frac{\mathrm{d}m}{\mathrm{d}t} = (1+\alpha)\frac{\mathrm{d}m_h}{\mathrm{d}t} \tag{2}$$

It is convenient to choose the total mass of the ejected propellant m_p as the new independent variable and reduce Eqs. (1) and (2) to

$$\frac{\mathrm{d}m_h}{\mathrm{d}m_p} = \frac{1}{(1+\alpha)} \tag{3}$$

$$\frac{\mathrm{d}V}{\mathrm{d}m_p} = \frac{c}{1-m_p} \tag{4}$$

(the rocket initial mass has been arbitrarily assumed as unitary, i.e., masses are expressed as fractions of the gross mass). The mixture ratio is not constant but is considered as the control variable of the problem, and the theory of optimal control is used to maximize the rocket performance.

The Hamiltonian

$$H = [\lambda_h/(1+\alpha)] + [\lambda_V c/(1-m_p)] \tag{5}$$

does not depend on the state variables; on the basis of the Euler–Lagrange equations, therefore, the adjoint variables λ_h and λ_V result as constant. Note that the process is not autonomous as H is an explicit function of the independent variable m_p.

The optimal value of the mixture ratio is provided by nullifying the partial derivative of the Hamiltonian with respect to α. One easily obtains

$$\frac{\partial H}{\partial \alpha} = \frac{\lambda_h}{(1+\alpha)^2} - \frac{\lambda_V c'}{1-m_p} = 0 \tag{6}$$

which can be solved for α when the function $c(\alpha)$ and its first derivative c' are known.

Boundary Conditions

The problem is completed by the boundary conditions. Initial values ($m_{pi} = 0$) for Eqs. (3) and (4) are $m_{hi} = V_i = 0$. As the final velocity is prescribed ($V_f = \Delta V$), two further conditions are needed to obtain the unknown parameters m_{pf}, λ_h, and λ_V. Because the final mass of the exhausted propellant is not specified, the necessary conditions for optimality are[6]

$$\lambda_h = \frac{\partial \varphi}{\partial m_{hf}} \tag{7}$$

$$H_f = -\frac{\partial \varphi}{\partial m_{pf}} \tag{8}$$

Two performance indexes φ are maximized: the former is related to the payload and the latter to the rocket dry mass. The maximization of the payload appears to be more important for high-ΔV operations, when the mission feasibility itself is uncertain. On the other hand, the minimization of the dry mass is more directly related to the mission cost.

The vehicle dry mass and propellant mass constitute the unitary gross mass. The dry mass

$$m_d = 1 - m_{pf} = m_u + m_t + m_e \tag{9}$$

is the sum of the payload m_u, the tank mass m_t, and the engine mass m_e, which also includes all other masses that do not change as the propulsion requirements change. The tank mass is linearly related to the propellant volume via a structural efficiency

$$m_t = (\epsilon/\rho_h)m_{hf} + (\epsilon/\rho_o)m_{of} \tag{10}$$

Maximum Payload

When the payload is maximized, the performance index is expressed, by means of Eq. (10), as a function of the independent and state variables

$$\varphi = m_u = 1 - m_e - m_t - m_{pf} = 1 - m_e - Am_{hf} - Bm_{pf} \tag{11}$$

where the parameters

$$A = (\epsilon/\rho_h) - (\epsilon/\rho_o) \tag{12}$$

$$B = 1 + (\epsilon/\rho_o) \tag{13}$$

only depend on tank and propellant characteristics. By using Eqs. (7) and (8), one obtains

$$\lambda_h = -A \tag{14}$$

$$\frac{\lambda_h}{1+\alpha_f} + \frac{\lambda_V c_f}{1-m_{pf}} = B \tag{15}$$

One obtains

$$A[c_f + c'_f(1+\alpha_f)] = -(1+\alpha_f)^2 B \tag{16}$$

by combining Eq. (15) with Eq. (6), when it is applied at the final point. The resulting Eq. (16) states that the final value of the mixture ratio depends only on propellant and tank characteristics and is not influenced by the required velocity increment or propellant consumption.

Maximum Payload/Dry-Mass Ratio

The performance index in the preceding subsection is actually the payload/gross-mass ratio; when the payload is assigned, the optimal strategy minimizes the gross mass. In a similar way, the minimization of the dry mass for an assigned payload is equivalent to the maximization of

$$\varphi = \frac{m_u}{m_d} = B - \frac{A\,m_{hf} + D}{1-m_{pf}} \tag{17}$$

where

$$D = (\epsilon/\rho_o) + m_e \tag{18}$$

Equations (7) and (8) provide

$$\lambda_h = -\frac{A}{1-m_{pf}} \tag{19}$$

$$\frac{\lambda_h}{1+\alpha_f} + \frac{\lambda_V c_f}{1-m_{pf}} = \frac{A\,m_{hf} + D}{(1-m_{pf})^2} \tag{20}$$

The final value of α now depends on the propellant consumption and, therefore, on the required ΔV.

In this case, the initial value α_i is independent of the characteristic velocity ΔV. By combining Eqs. (19) and (20), the ratio

$$\frac{\lambda_V}{\lambda_h} = -\frac{1}{c_f}\left(\frac{1-m_{pf}}{1+\alpha_f} + m_{hf} + \frac{D}{A}\right) \tag{21}$$

seems to be a function of the propellant consumption. During the optimal rocket operations, however, m_h and α are related to m_p via Eqs. (3) and (6). If these relations are taken into account, the total derivative of Eq. (21) with respect to m_{pf} is found to be zero, i.e., λ_V/λ_h is actually independent of the propellant consumption. Therefore, a single value α_i (that does not depend on ΔV) is furnished by Eq. (6) once the tank, propellant, and rocket characteristics (ϵ, ρ, m_e) are given. Thereafter, the control and in turn the state variables are univocally determined by the value of the independent variable.

Numerical Results

The application of OCT produces a boundary-value problem that must be numerically solved. The authors have used an efficient procedure, which they developed for the optimization of spacecraft maneuvers[7]; however, a large variety of analogous codes is available. Numerical computations have been carried out by assuming $\epsilon = 16$ kg/m^4 and $m_e = 0.05$, according to Martin's[8] suggestions (m_e is not identically defined). The effective exhaust velocity in a vacuum has been evaluated (Fig. 1) by assuming 200 bar in the combustion chamber and one-dimensional frozen flow in an ideal nozzle with an area ratio of 100. For the sake of simplicity, the dependance on the mixture ratio is approximated by means of the parabolic relation

$$c = c_0 - c_1(\alpha - \alpha^*)^2 \tag{22}$$

where $\alpha^* = 3.8$, $c_0 = 0.5905$, and $c_1 = 0.00408$ (throughout velocities are normalized by using the circular velocity at the Earth surface as the reference value). A numerical solution can, however, be obtained for any different function $c(\alpha)$.

Maximum Payload

The optimal control laws are presented in Fig. 2 for different values of the ideal velocity increment or mission characteristic velocity. The mixture ratio is progressively reduced while the propellants are consumed; according to Eq. (16) the final mixture ratio is always the same. The constancy of the minimum mixture ratio is also shown in Fig. 3, which presents the minimum, mean, and maximum values of the mixture ratio as a function of the velocity increment; the best value α_c for constant mixture-ratio operations, which is derived in the Appendix, is also presented. Note that α_c increases with the mission characteristic velocity and moves far from the value α^*,

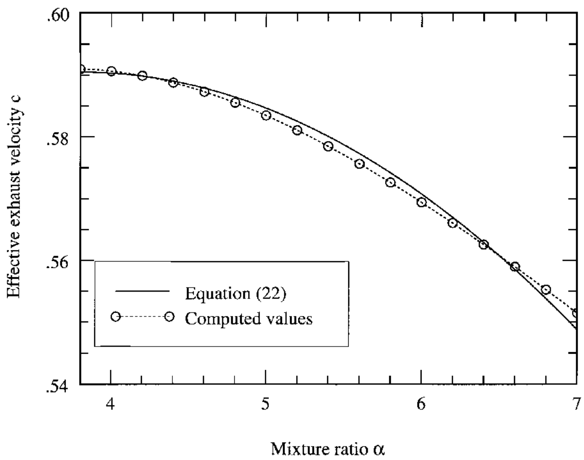

Fig. 1 Influence of the mixture ratio on the effective exhaust velocity.

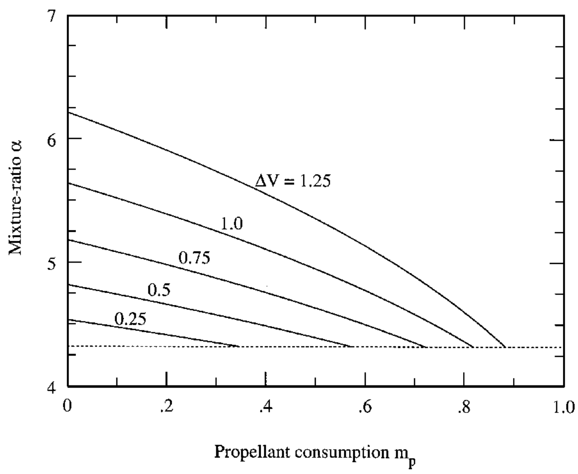

Fig. 2 Control laws for maximum payload.

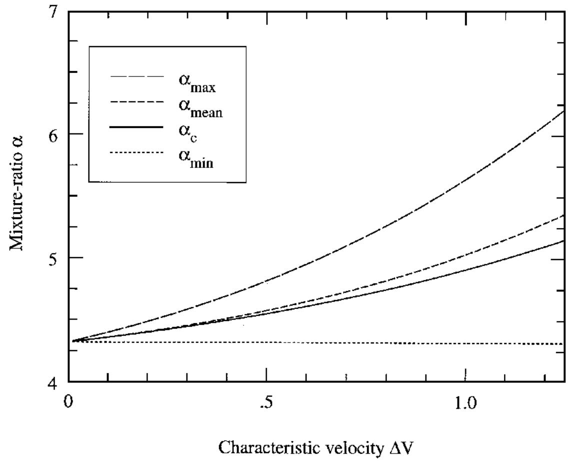

Fig. 3 Mixture ratios for maximum payload.

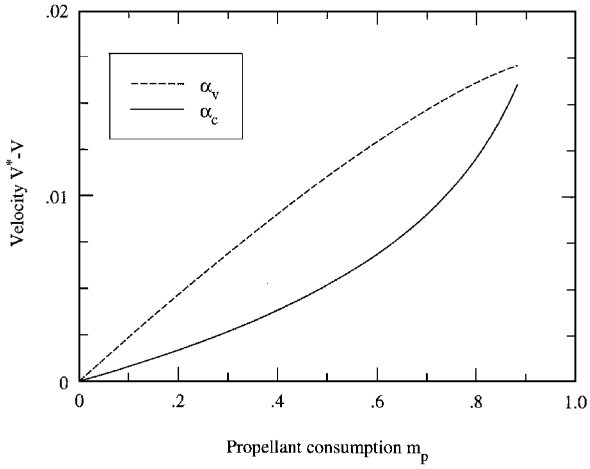

Fig. 4 Rocket velocity in comparison to maximum-*c* operation: maximum payload, ΔV=1.25.

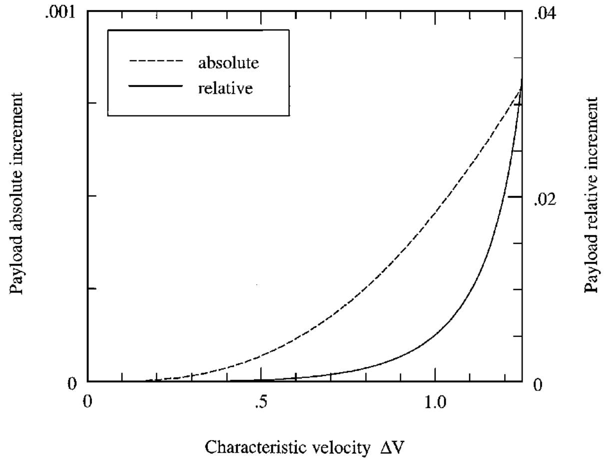

Fig. 5 Benefit of the mixture-ratio control: maximum payload.

which corresponds to the maximum specific impulse; in the case of demanding missions, a high propellant bulk density is preferred to a high specific impulse (and the list of attractive propellants again includes RP-1).

The mean mixture ratio for the optimal operations is higher than α_c (i.e., the oxygen consumption is enhanced). The optimal control of the engine improves the rocket overall performance instead of the engine performance; the rocket acceleration is less vigorous but the tank mass is advantageously reduced. Figure 4 compares the velocities that are achieved using either the optimal or the constant mixture ratio to the velocity V^*, which is attained for the same propellant consumption by operating with the maximum specific impulse.

Figure 5 highlights the benefit of the optimal control with respect to the constant mixture-ratio operations, in terms of useful mass vs

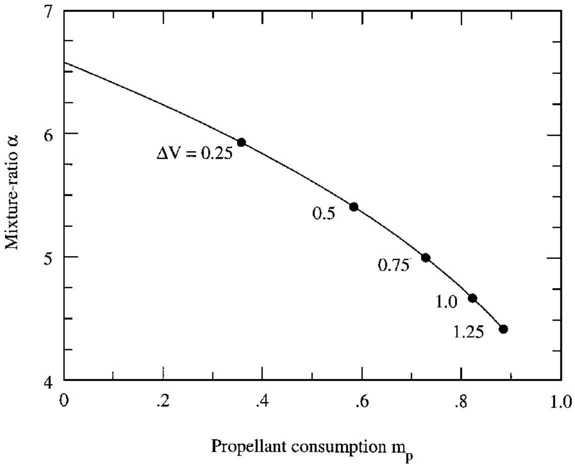

Fig. 6 Control law for minimum dry mass.

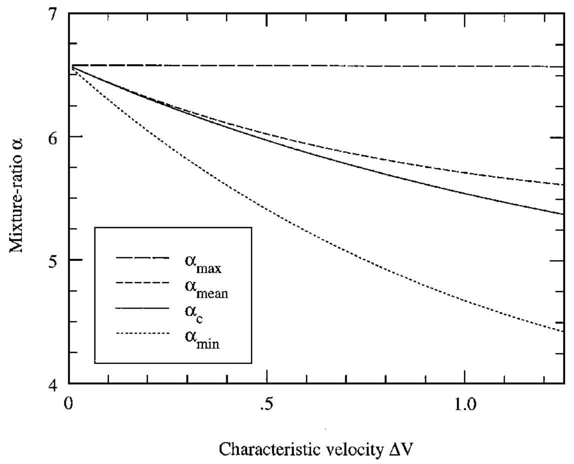

Fig. 7 Mixture ratios for minimum dry mass.

the ideal velocity increment; the mixture-ratio control is appealing for the most demanding (or marginal) missions. Note that the initial operation at a high mixture ratio permits a high thrust-to-weight ratio, which in turn improves the ascent trajectory. This further benefit is not included in Fig. 5, which only takes the influence of the propellant bulk density into account.

Maximum Payload/Dry-Mass Ratio

Figures 6 and 7 refer to the minimization of the rocket dry mass for an assigned payload. It has been shown, by means of Eq. (21), that the optimal mixture ratio does not depend on ΔV but that engines operate according to the same control law until the final velocity is achieved. In this case, when the mission becomes more demanding, the best constant mixture ratio moves toward the maximum specific impulse but the optimal variable-α solution still uses more oxidizer than the best constant-α solution.

A comparison of the results suggests that the use of either performance index is equivalent for high-ΔV missions. In contrast, the control law is quite different if the velocity increment is moderate.

Conclusion

A rather simple model has been used to roughly quantify the improvement of rocket performance that is produced by the optimal control of the mixture ratio. The control law for low-ΔV missions is quite different when the payload or the dry mass is assumed as the performance index. The variation of the mixture ratio during the engine operation, however, is modest, and similar performance is obtained by using a constant mixture ratio. In the case of demanding missions, the optimal control is almost the same, independent of the performance index considered. Moreover, the scope of the mixture ratio is wide, i.e., quite far from a constant value, and variable mixture ratio engines perform significantly better than constant mixture ratio engines. The actual benefit is probably larger because the influence of the mixture ratio on the thrust level has not been taken into account, whereas the reduction of gravitational losses demands a larger amount of oxidizer in the initial phase of the ascent. In conclusion, when a mission is marginal for current technology (for instance, the exploitation of an SSTO rocket system), mixture-ratio control should be considered as one of the techniques that could make the mission feasible.

Appendix: Constant Mixture-Ratio Operation

The rocket operation with constant mixture ratio is considered. In this case, Eq. (10) is more conveniently expressed in the form

$$m_t = [m_{pf}/(1+\alpha)][(\epsilon/\rho_h) + (\epsilon/\rho_o)\alpha] = E\, m_{pf} \tag{A1}$$

and one easily obtains

$$\frac{\mathrm{d}E}{\mathrm{d}\alpha} = -\frac{A}{(1+\alpha)^2} \tag{A2}$$

using parameter A as defined by Eq. (12).

Because c is constant, the analytical integration of Eq. (4) produces the Tsiolkovsky equation[2]

$$m_{pf} = 1 - \exp[-(\Delta V)/c] \tag{A3}$$

thereafter,

$$\frac{\mathrm{d}m_{pf}}{\mathrm{d}\alpha} = -\frac{\Delta V}{c} c'(1 - m_{pf}) \tag{A4}$$

The optimal value α_c of the constant mixture ratio is obtained by nullifying the derivative of the performance index with respect to α.

Maximum Payload

Equation (11) is posed in the form

$$\varphi = 1 - m_e - (1+E)m_{pf} \tag{A5}$$

and the equation

$$(1+E)\frac{\mathrm{d}m_{pf}}{\mathrm{d}\alpha} + \frac{\mathrm{d}E}{\mathrm{d}\alpha} m_{pf} = 0 \tag{A6}$$

is numerically solved for α_c, using Eqs. (A2) and (A4).

Maximum Payload/Dry-Mass Ratio

In the case of constant mixture-ratio operations, Eq. (17) becomes

$$\varphi = 1 - \frac{m_e + E m_{pf}}{1 - m_{pf}} \tag{A7}$$

and α_c is obtained from

$$\frac{m_e + E}{1 - m_{pf}} \frac{\mathrm{d}m_{pf}}{\mathrm{d}\alpha} + \frac{\mathrm{d}E}{\mathrm{d}\alpha} m_{pf} = 0 \tag{A8}$$

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